Rates of convergence in the strong invariance principle for non adapted sequences. Application to ergodic automorphisms of the torus

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Abstract

In this paper, we give rates of convergence in the strong invariance principle for non-adapted sequences satisfying projective criteria. The results apply to the iterates of ergodic automorphisms T of the d-dimensional torus $\mathbb{T}^d$, even in the non hyperbolic case. In this context, we give a large class of unbounded function f from $\mathbb{T}^d$ to $\mathbb{R}$, for which the partial sum $f \circ T + f \circ T^2 + \cdots + f \circ T^n$ satisfies a strong invariance principle with an explicit rate of convergence.

1. Introduction and notations

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $T : \Omega \mapsto \Omega$ be a bijective bimeasurable transformation preserving the probability $\mathbb{P}$. For a σ-algebra $\mathcal{F}_0$ satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$, we define the nondecreasing filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ by $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$. The $\mathbb{L}^p$ norm of a random variable X is denoted by $\|X\|_p = (\mathbb{E}(|X|^p))^{1/p}$.

Let X_0 be a real-valued and square integrable random variable such that $\mathbb{E}(X_0) = 0$, and define the stationary sequence $(X_i)_{i\in\mathbb{Z}}$ by $X_i = X_0 \circ T^i$. Define then the partial sum by $S_n = X_1 + X_2 + \cdots + X_n$. According to the Birkhoff-Khinchine theorem, S_n satisfies a strong law of large numbers. One can go further in the study of the statistical properties of S_n. We study here the rate of convergence in the almost sure invariance principle (ASIP). More precisely, we give conditions under which there exists a sequence of independent identically

distributed (iid) Gaussian random variables $(Z_i)_{i\geq 1}$ such that

$$\sup_{1\leq k\leq n}\Big|\sum_{i=1}^{k}(X_i - Z_i)\Big| = o(n^{1/p}L(n)) \quad \text{almost surely}, \tag{1.1}$$

for $p \in]2,4]$ and L an explicit slowly varying function. Let us recall that, in the iid case, Komlós, Major and Tusnády [11] and Major [16] obtained an ASIP with the optimal rate $o(n^{1/p})$ in (1.1) as soon as the random variables admit a moment of order p.

Since the seminal paper by Philipp and Stout [23], many authors have considered this problem in a dependent context, but most of the papers deal with the adapted case, when X_0 is $\mathcal{F}_0$ measurable (for instance, $\mathcal{F}_0$ is the past σ-algebra $\sigma(X_i, i\leq 0)$). Unfortunately, it is quite common to encounter dynamical systems for which the natural filtration does not allow to control any quantity involving terms of the type $\|\mathbb{E}(X_n|\mathcal{F}_0)\|_p$.

In this paper, we shall not assume that X_0 is $\mathcal{F}_0$-measurable, and we shall give conditions on $\|\mathbb{E}(X_n|\mathcal{F}_0)\|_p$, $\|X_{-n} - \mathbb{E}(X_{-n}|\mathcal{F}_0)\|_p$ and $\|\mathbb{E}(S_n^2|\mathcal{F}_{-n}) - \mathbb{E}(S_n^2)\|_{p/2}$ for (1.1) to hold (see Theorems 3.1 and 3.2 of Section 3). These conditions are in the same spirit as those given by Gordin [6] for $p = 2$ to get the usual central limit theorem. Our proof is based on the approximation

$$\sum_{i=1}^{n} X_i = M_n + R_n$$

by the martingale $M_n = d_1 + d_2 + \cdots + d_n$, where d_i is the martingale difference

$$d_i = \sum_{k\in\mathbb{Z}}(\mathbb{E}(X_k|\mathcal{F}_i) - \mathbb{E}(X_k|\mathcal{F}_{i-1}))$$

introduced by Gordin [6] and Heyde [9]. In the adapted case, similar conditions are given in the recent paper [1], together with a long list of applications.

In the non adapted case, it is easy to see that our results apply to a large class of two-sided functions of iid sequences, or two-sided functions of absolutely regular sequences. But they also apply to much complicated dynamical systems, for which such a representation by functions of absolutely regular sequences is not available. In the next section, we consider the case where T is an ergodic automorphism of the d-dimensional torus $\mathbb{T}^d$, and $\mathbb{P}$ is the Lebesgue measure on $\mathbb{T}^d$. In this context, we use the σ-algebra $\mathcal{F}_i$ considered by Le Borgne [12]. As a consequence of Theorem 2.1, we obtain that (1.1) holds for $p = 4$ and $X_i = f\circ T^i$, where $f : \mathbb{T}^d \to \mathbb{R}$, as soon as the Fourier coefficients $(c_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ of f are such that

$$|c_{\mathbf{k}}| \leq A\prod_{i=1}^{d}\frac{1}{(1+|k_i|)^{3/4}\log^{\alpha}(2+|k_i|)} \quad \text{for some } \alpha > 13/8.$$

We also get that there exists a positive ε such that

$$\sup_{1\leq k\leq n}\Big|\sum_{i=1}^{k}(X_i - Z_i)\Big| = o(n^{1/2-\varepsilon}) \quad \text{almost surely},$$

as soon as

$$|c_{\mathbf{k}}| \leq A\prod_{i=1}^{d}\frac{1}{(1+|k_i|)^{\delta}} \quad \text{for some } \delta > 1/2.$$

These rates of convergence in the almost sure invariance principle complement the results by Leonov [14] and Le Borgne [12] for the central limit theorem and the almost sure invariance

principle respectively. Let us mention that Dolgopyat [4] established an ASIP with the rate $o(n^{1/2-\varepsilon})$ (for some $\varepsilon > 0$) valid for ergodic automorphisms of the torus and f a Hölder continuous function. Thanks to the decorrelation estimates obtained in [13], the rate for Hölder observables can be improved by applying the general result of Gouëzel in [7] to get the rate $o(n^{1/4+\varepsilon})$ for every $\varepsilon > 0$, and by applying the results of the present paper to get the rate $o(n^{1/4}L(n))$. Up to our knowledge, the present work gives the first strong approximations results for such partially hyperbolic transformations T for unbounded (and then non continuous) functions f.

To conclude, let us mention some previous works in the context of dynamical systems: several results have been established with the rate $o(n^{1/2-\varepsilon})$ for some $\varepsilon > 0$ (see [10, 3, 4, 22, 17]). Results giving a rate in $o(n^{1/4+\varepsilon})$ for every $\varepsilon > 0$ can be found in [19, 5, 18, 7]. Most of these results hold for bounded functions f.

Let us precise once again that we can reach the rate $o(n^{1/4}L(n))$ instead of $o(n^{1/4+\varepsilon})$ for every $\varepsilon > 0$. Moreover, our conditions giving the rate $o(n^{1/p}L(n))$ are related to moments of order p of f. Such results are not very common in the context of dynamical systems (let us mention [7] in the particular case of Gibbs-Markov maps, and [2, 21] for generalized Pommeau-Manneville maps).

2. ASIP with rates for ergodic automorphisms of the torus

Let $d \geq 2$. We consider a group automorphism T of the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. For every $x \in \mathbb{R}^d$, we write $\bar{x}$ its class in $\mathbb{T}^d$. We recall that T is the quotient map of a linear map $\tilde{T} : \mathbb{R}^d \to \mathbb{R}^d$ given by $\tilde{T}(x) = S \cdot x$, where S is a $d \times d$-matrix with integer entries and with determinant 1 or -1. The map $x \mapsto S \cdot x$ preserves the infinite Lebesgue measure λ on $\mathbb{R}^d$ and T preserves the probability Lebesgue measure $\bar{\lambda}$. We suppose T ergodic, which is equivalent to the fact that no eigenvalue of S is a root of the unity. In this case, it is known that the spectral radius of S is larger than one (and so S admits at least an eigenvalue of modulus larger than one and at least an eigenvalue of modulus smaller than one). This hypothesis holds true in the case of hyperbolic automorphisms of the torus (i.e. in the case when no eigenvalue of S has modulus one) but is much weaker. Indeed, as mentioned in [12], the following matrix gives an example of an ergodic non hyperbolic automorphism of $\mathbb{T}^4$:

$$S := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

When T is ergodic and non hyperbolic, the dynamical system $(\mathbb{T}^d, T, \bar{\lambda})$ has no Markov partition. However, it is possible to construct some measurable partition [15], to prove a central limit theorem [14]. Moreover, in [12], Le Borgne proved the functional central limit theorem and the Strassen strong invariance principle for $(X_k = f \circ T^k)_k$ under weak hypotheses on f, thanks to Gordin's method and to the partitions studied by Lind in [15].

We give here rates of convergence in the strong invariance principle for $(X_k = f \circ T^k)_k$ under conditions on the Fourier coefficients of $f : \mathbb{T}^d \to \mathbb{R}$. In what follows, for $\mathbf{k} \in \mathbb{Z}^d$, we denote by $|\mathbf{k}| = \max_{i \in \{1,\dots,d\}} |k_i|$.

Theorem 2.1. *Let T be an ergodic automorphism of $\mathbb{T}^d$ with the notations as above. Let $p \in]2, 4]$ and q be its conjugate exponent. Let $f : \mathbb{T}^d \to \mathbb{R}$ be a centered function with Fourier*

coefficients $(c_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ *satisfying, for any integer* $b \geq 2$,

$$\sum_{|\mathbf{k}|\geq b} |c_{\mathbf{k}}|^q \leq R \log^{-\theta}(b) \quad \textit{for some } \theta > \frac{p^2-2}{p(p-1)}\,, \tag{2.2}$$

and

$$\sum_{|\mathbf{k}|\geq b} |c_{\mathbf{k}}|^2 \leq R \log^{-\beta}(b) \quad \textit{for some } \beta > \frac{3p-4}{p}\,. \tag{2.3}$$

Then the series

$$\sigma^2 = \bar\lambda((f-\bar\lambda(f))^2) + 2\sum_{k>0} \bar\lambda((f-\bar\lambda(f)) f\circ T^k)$$

converges absolutely and, enlarging $\mathbb{T}^d$ *if necessary, there exists a sequence* $(Z_i)_{i\geq 1}$ *of iid gaussian random variables with zero mean and variance* σ^2 *such that, for any* $t > 2/p$,

$$\sup_{1\leq k\leq n} \Big| \sum_{i=1}^k f\circ T^i - \sum_{i=1}^k Z_i \Big| = o\big(n^{1/p}(\log n)^{(t+1)/2}\big) \textit{ almost surely, as } n\to\infty. \tag{2.4}$$

Observe that (2.3) follows from (2.2) provided that $\theta > (3p-4)/(2p-2)$. Hence, (2.2) and (2.3) are both satisfied as soon as

$$\sum_{|\mathbf{k}|\geq b} |c_{\mathbf{k}}|^q \leq R \log^{-\theta}(b) \quad \text{for some } \theta > \frac{3p-4}{2(p-1)}\,.$$

Let us now compare our hypotheses on Fourier coefficients with those appearing in other works. In [14], Leonov proved a central limit theorem (possibly degenerated) when

$$|c_{\mathbf{k}}| \leq A \prod_{i=1}^d \frac{1}{(1+|k_i|)^{1/2}\log^{\alpha}(2+|k_i|)} \quad \text{for some } \alpha > 3/2. \tag{2.5}$$

In [12], Le Borgne proved the functional central limit theorem and the Strassen strong invariance principle when (2.3) holds true with $\beta > 2$ (and when f is not a coboundary), which is a weaker condition than (2.5). Observe that, as p converges to 2, $(p^2-2)/(p(p-1))$ and $(3p-4)/p$ both converge to 1.

3. Probabilistic results

In the rest of the paper, we shall use the following notations: $\mathbb{E}_k(X) = \mathbb{E}(X|\mathcal{F}_k)$, and $a_n \ll b_n$ means that there exists a numerical constant C not depending on n such that $a_n \leq C b_n$, for all positive integers n.

In this section, we give rates of convergence in the strong invariance principle under projective criteria for stationary sequences that are non necessarily adapted to $\mathcal{F}_i$.

Theorem 3.1. *Let* $2 < p < 4$ *and* $t > 2/p$. *Assume that* X_0 *belongs to* $\mathbb{L}^p$, *that*

$$\sum_{n\geq 2} \frac{n^{p-1}}{n^{2/p}(\log n)^{(t-1)p/2}} \big(\|\mathbb{E}_0(X_n)\|_p^p + \|X_{-n} - \mathbb{E}_0(X_{-n})\|_p^p\big) < \infty\,, \tag{3.1}$$

and that

$$\sum_{n\geq 2} \frac{n^{3p/4}}{n^2(\log n)^{(t-1)p/2}} \big(\|\mathbb{E}_0(X_n)\|_2^{p/2} + \|X_{-n} - \mathbb{E}_0(X_{-n})\|_2^{p/2}\big) < \infty\,. \tag{3.2}$$

Assume in addition that there exists a positive integer m such that

$$\sum_{n\geq 2}\frac{1}{n^2(\log n)^{(t-1)p/2}}\big\|\mathbb{E}_{-nm}(S_n^2)-\mathbb{E}(S_n^2)\big\|_{p/2}^{p/2}<\infty\,. \tag{3.3}$$

Then $n^{-1}\mathbb{E}(S_n^2)$ converges to $\sigma^2=\sum_{k\in\mathbb{Z}}\mathrm{Cov}(X_0,X_k)$ and, enlarging Ω if necessary, there exists a sequence $(Z_i)_{i\geq 1}$ of iid Gaussian random variables with zero mean and variance σ^2 such that

$$\sup_{1\leq k\leq n}\Big|S_k-\sum_{i=1}^k Z_i\Big|=o\big(n^{1/p}(\log n)^{(t+1)/2}\big)\quad \textit{almost surely, as } n\to\infty. \tag{3.4}$$

Theorem 3.2. *Let $t>1/2$. Assume that X_0 belongs to $\mathbb{L}^4$ and that the conditions (3.1) and (3.3) hold with $p=4$. Assume in addition that*

$$\sum_{n\geq 2}n(\log n)^{4-2t}\big(\|\mathbb{E}_0(X_n)\|_2^2+\|X_{-n}-\mathbb{E}_0(X_{-n})\|_2^2\big)<\infty\,. \tag{3.5}$$

Then the conclusion of Theorem 3.1 holds with $p=4$.

Proof of Theorems 3.1 and 3.2. We first notice that since $p>2$, (3.1) implies that

$$\sum_{n>0}n^{-1/p}\|\mathbb{E}_0(X_n)\|_p<\infty \ \text{ and } \ \sum_{n>0}n^{-1/p}\|X_{-n}-\mathbb{E}_0(X_{-n})\|_p<\infty$$

(apply Hölder's inequality to see this). Let $P_k(X)=\mathbb{E}_k(X)-\mathbb{E}_{k-1}(X)$. Using Lemma 5.1 of the appendix with $q=1$, we infer that

$$\sum_{k\in\mathbb{Z}}\|P_0(X_k)\|_p<\infty\,. \tag{3.6}$$

In addition (3.6) implies that $n^{-1}\mathbb{E}(S_n^2)$ converges to $\sigma^2=\sum_{k\in\mathbf{Z}}\mathrm{Cov}(X_0,X_k)$.

Let now $d_0:=\sum_{j\in\mathbb{Z}}P_0(X_j)$. Then d_0 belongs to $\mathbb{L}^p$ and satisfies $\mathbb{E}(d_0|\mathcal{F}_{-1})=0$. Let $d_i:=d_0\circ T^i$ for all $i\in\mathbb{Z}$. Then $(d_i)_{i\in\mathbb{Z}}$ is a stationary sequence of martingale differences in $\mathbb{L}^p$. Let

$$M_n:=\sum_{i=1}^n d_i \ \text{ and } \ R_n:=S_n-M_n\,.$$

The theorems will be proven if we can show that

$$R_n=o\big(n^{1/p}(\log n)^{(t+1)/2}\big)\quad\text{almost surely as } n\to\infty, \tag{3.7}$$

and that (3.4) holds true with M_k replacing S_k. Since $\mathbb{E}(d_0^2)=\sigma^2$ and $t>p/2$, according to Proposition 5.1 in [1] (applied with $\psi(n):=n^{2/p}(\log n)^t$), to prove that (3.4) holds true with M_k replacing S_k, it suffices to prove that

$$\sum_{n\geq 2}\frac{1}{n^2(\log n)^{(t-1)p/2}}\big\|\mathbb{E}_0(M_n^2)-\mathbb{E}(M_n^2)\big\|_{p/2}^{p/2}<\infty\,. \tag{3.8}$$

By standard arguments, (3.7) will be satisfied if we can show that

$$\sum_{r>0}\frac{\|\max_{1\leq \ell\leq 2^r}|R_\ell|\|_p^p}{2^r\, r^{(t+1)p/2}}<\infty\,. \tag{3.9}$$

Now, by stationarity, $\| \max_{1\le \ell \le 2^r} |R_\ell| \|_p \ll 2^{r/p} \sum_{k=0}^r 2^{-k/p} \|R_{2^k}\|_p$ (see for instance inequality (6) in [24]) and for all $i,j \geq 0$, $\|R_{i+j}\|_q \leq \|R_i\|_q + \|R_j\|_q$. Applying then Item 1 of Lemma 37 in [20], we derive that for any integer n in $[2^r, 2^{r+1}[$,

$$\Big\| \max_{1\le \ell \le 2^r} |R_\ell| \Big\|_p \ll n^{1/p} \sum_{k=1}^n k^{-(1+1/p)} \|R_k\|_p \,. \tag{3.10}$$

Therefore using (3.10) followed by an application of Hölder's inequality, we get that for any $\alpha < 1$,

$$\begin{aligned} \sum_{r>0} \frac{\| \max_{1\le \ell \le 2^r} |R_\ell| \|_p^p}{2^r\, r^{(t+1)p/2}} &\ll \sum_{n \geq 2} \frac{1}{n\,(\log n)^{(t+1)p/2}} \Big(\sum_{k=1}^n k^{-(1+1/p)} \|R_k\|_p \Big)^p \\ &\ll \sum_{n\geq 2} \frac{(\log n)^{(p-1)(1-\alpha)}}{n\,(\log n)^{(t+1)p/2}} \sum_{k=1}^n k^{-2} (\log k)^{\alpha(p-1)} \|R_k\|_p^p \,. \end{aligned}$$

Hence taking $\alpha \in]1 - p/(2(p-1)), 1[$ and changing the order of summation, we infer that (3.9) and then (3.7) hold provided that

$$\sum_{n \geq 1} \frac{\|R_n\|_p^p}{n^2 (\log n)^{(t-1)p/2}} < \infty \,. \tag{3.11}$$

On an other hand, we shall prove that condition (3.8) is implied by: there exists a positive finite integer m such that

$$\sum_{n\geq 2} \frac{1}{n^2 (\log n)^{(t-1)p/2}} \| \mathbb{E}_{-nm}(M_n^2) - \mathbb{E}(M_n^2) \|_{p/2}^{p/2} < \infty \,. \tag{3.12}$$

For any nonnegative integer i, we set $V_i := \| \mathbb{E}_0(M_i^2) - \mathbb{E}(M_i^2) \|_{p/2}$. Using that M_n is a martingale, we infer that, for any nonnegative integers i and j,

$$V_{i+j} \leq V_i + V_j \,. \tag{3.13}$$

Let now $n \in [2^k, 2^{k+1} - 1] \cap \mathbb{N}$, and write its binary expansion:

$$n = \sum_{\ell=0}^k 2^\ell b_\ell \quad \text{where } b_k = 1 \text{ and } b_j \in \{0,1\} \text{ for } j = 0, \dots, k-1 \,.$$

Inequality (3.13) combined with Hölder's inequality implies that, for any $\eta > 0$,

$$V_n^{p/2} \leq \Big(\sum_{\ell=0}^k V_{2^\ell} \Big)^{p/2} \ll 2^{\eta p (k+1)/2} \sum_{\ell=0}^k \Big(\frac{V_{2^\ell}}{2^{\eta \ell}} \Big)^{p/2} \,. \tag{3.14}$$

Therefore

$$\sum_{n\geq 2} \frac{1}{n^2 (\log n)^{(t-1)p/2}} V_n^{p/2} \ll \sum_{k>0} \frac{2^{\eta p (k+1)/2}}{2^k k^{(t-1)p/2}} \sum_{\ell=0}^k \Big(\frac{V_{2^\ell}}{2^{\eta \ell}} \Big)^{p/2} \,.$$

Changing the order of summation and taking $\eta \in]0, 2/p[$, it follows that (3.8) is implied by

$$\sum_{k \geq 1} \frac{1}{2^k k^{(t-1)p/2}} \| \mathbb{E}_0(M_{2^k}^2) - \mathbb{E}(M_{2^k}^2) \|_{p/2}^{p/2} < \infty \tag{3.15}$$

(actually due to the subadditivity of the sequence (V_i) both conditions are equivalent, see the proof of item 1 of Lemma 37 in [20] to prove that (3.8) entails (3.15)). Now, since (M_n) is a martingale,

$$\mathbb{E}_0(M_{2^k}^2) - \mathbb{E}(M_{2^k}^2) = \sum_{j=1}^{k} \big(\mathbb{E}_0((M_{2^j} - M_{2^{j-1}})^2) - \mathbb{E}((M_{2^j} - M_{2^{j-1}})^2) \big) + \mathbb{E}_0(d_1^2) - \mathbb{E}(d_1^2) ,$$

which implies by stationarity that

$$\big\| \mathbb{E}_0(M_{2^k}^2) - \mathbb{E}(M_{2^k}^2) \big\|_{p/2} \leq \sum_{j=0}^{k-1} \big\| \mathbb{E}_{-2^j}(M_{2^j}^2) - \mathbb{E}(M_{2^j}^2) \big\|_{p/2} + \big\| \mathbb{E}_0(d_1^2) - \mathbb{E}(d_1^2) \big\|_{p/2} .$$

Therefore by using Hölder's inequality as done in (3.14) with $\eta \in]0, 2/p[$, we infer that (3.15) is implied by

$$\sum_{k \geq 1} \frac{1}{2^k k^{(t-1)p/2}} \big\| \mathbb{E}_{-2^k}(M_{2^k}^2) - \mathbb{E}(M_{2^k}^2) \big\|_{p/2}^{p/2} < \infty . \tag{3.16}$$

Notice now that the sequence $(W_n)_{n>0}$ defined by

$$W_n := \big\| \mathbb{E}_{-n}(M_n^2) - \mathbb{E}(M_n^2) \big\|_{p/2}$$

is subadditive. Indeed, for any non negative integers i and j, using that M_n is a martingale together with the stationarity, we derive that

$$\begin{aligned} W_{i+j} &= \big\| \mathbb{E}_{-(i+j)}(M_i^2) - \mathbb{E}(M_i^2) + \mathbb{E}_{-(i+j)}((M_{i+j} - M_i)^2) - \mathbb{E}((M_{i+j} - M_i)^2) \big\|_{p/2} \\ &\leq \big\| \mathbb{E}_{-i}(M_i^2) - \mathbb{E}(M_i^2) \big\|_{p/2} + \big\| \mathbb{E}_{-j}(M_j)^2) - \mathbb{E}(M_j)^2) \big\|_{p/2} \\ &\leq W_i + W_j . \end{aligned}$$

Therefore $W_{i+j}^{p/2} \leq 2^{p/2} W_i^{p/2} + 2^{p/2} W_j^{p/2}$. This implies that, for any integer ℓ and any integer $0 \leq j \leq \ell$,

$$W_\ell^{p/2} \leq 2^{p/2}(W_j^{p/2} + W_{\ell-j}^{p/2}), \text{ in such a way that } (\ell+1) W_\ell^{p/2} \leq 2^{1+p/2} \sum_{j=1}^{\ell} W_j^{p/2} . \tag{3.17}$$

Therefore using the second part of (3.17) with $\ell = 2^k$, we infer that condition (3.16) is implied by

$$\sum_{n \geq 2} \frac{1}{n^2 (\log n)^{(t-1)p/2}} \big\| \mathbb{E}_{-n}(M_n^2) - \mathbb{E}(M_n^2) \big\|_{p/2}^{p/2} < \infty . \tag{3.18}$$

It remains to prove that (3.12) implies (3.18). With this aim, we have, for any positive integer m,

$$M_n = \sum_{k=1}^{m} \big(M_{k[nm^{-1}]} - M_{(k-1)[nm^{-1}]} \big) + M_n - M_{m[nm^{-1}]} .$$

Using that M_n is a martingale together with the stationarity, we then infer that

$$\begin{aligned} \big\| \mathbb{E}_{-n}(M_n^2) - \mathbb{E}(M_n^2) \big\|_{p/2}^{p/2} &\leq 2^{p/2} m^{p/2} \big\| \mathbb{E}_{-n}(M_{[nm^{-1}]}^2) - \mathbb{E}(M_{[nm^{-1}]}^2) \big\|_{p/2}^{p/2} \\ &\quad + 2^{p/2} \big\| \mathbb{E}_{-n}(M_{n-m[nm^{-1}]}^2) - \mathbb{E}(M_{n-m[nm^{-1}]}^2) \big\|_{p/2}^{p/2} , \end{aligned}$$

which, together with the fact that $n - m[nm^{-1}] < m$, implies that

$$\big\|\mathbb{E}_{-n}(M_n^2) - \mathbb{E}(M_n^2)\big\|_{p/2}^{p/2} \le 2^{p/2} m^{p/2}\Big(2^{p/2}\|d_0\|_p^p + \big\|\mathbb{E}_{-n}(M_{[nm^{-1}]}^2) - \mathbb{E}(M_{[nm^{-1}]}^2)\big\|_{p/2}^{p/2}\Big)$$
$$\le 2^{p/2} m^{p/2}\Big(2^{p/2}\|d_0\|_p^p + \big\|\mathbb{E}_{-m[nm^{-1}]}(M_{[nm^{-1}]}^2) - \mathbb{E}(M_{[nm^{-1}]}^2)\big\|_{p/2}^{p/2}\Big), \qquad (3.19)$$

where for the last line we have used the fact that $n \ge m[nm^{-1}]$. We notice now that due to the martingale property of (M_n) and to stationarity, the sequence $(U_i)_{i\ge 0}$ defined for any non negative integer i by

$$U_i := \big\|\mathbb{E}_{-mi}(M_i^2) - \mathbb{E}(M_i^2)\big\|_{p/2}^{p/2}$$

satisfies, for any positive integers i and j,

$$U_{i+j} \le \Big(\big\|\mathbb{E}_{-m(i+j)}(M_i^2) - \mathbb{E}(M_i^2)\big\|_{p/2} + \big\|\mathbb{E}_{-m(i+j)}((M_{i+j} - M_i)^2) - \mathbb{E}((M_{i+j} - M_i)^2)\big\|_{p/2}\Big)^{p/2} \le 2^{p/2} U_i + 2^{p/2} U_j\,.$$

Hence by (3.17) applied with $W_i^{p/2} = U_i$,

$$U_{[nm^{-1}]} \le 2^{1+p/2}([nm^{-1}] + 1)^{-1} \sum_{k=1}^{[nm^{-1}]} U_k \le 2^{1+p/2} \sum_{k=1}^{[nm^{-1}]} \frac{U_k}{k}\,. \qquad (3.20)$$

Therefore starting from (3.19), considering (3.20) and changing the order of summation, we infer that (3.18) (and so (3.8)) holds provided that (3.12) does. To end the proof, it remains to show that under the conditions of Theorems 3.1 and 3.2, the conditions (3.11) and (3.12) are satisfied. This is achieved by using the two following lemmas.

Lemma 3.1. *Let $p \in [2,4]$. Assume that (3.1) holds. Then*

$$\sum_{n\ge 1} \frac{\max_{1\le \ell\le n} \|R_\ell\|_p^p}{n^2(\log n)^{(t-1)p/2}} < \infty\,,$$

and (3.11) holds.

Lemma 3.2. *Let $p \in [2,4]$ and assume that (3.1) and (3.3) are satisfied. Assume in addition that (3.2) holds when $2 < p < 4$ and (3.5) does when $p = 4$. Then* (3.12) *is satisfied.*

It remains to prove the two above lemmas.

Proof of Lemma 3.1. Since (3.1) implies (3.6), Item 2 of Proposition 5.1 given in the appendix implies that, for any positive integers ℓ and N,

$$\|R_\ell\|_p \ll \max_{k=\ell,N} \|\mathbb{E}_0(S_k)\|_p + \max_{k=\ell,N} \|S_k - \mathbb{E}_k(S_k)\|_p + \ell^{1/2} \sum_{|j|\ge N} \|P_0(X_j)\|_p\,.$$

Next, applying Lemma 5.1 given in the appendix with $q = 1$, and using the fact that by stationarity, for any positive integer k,

$$\|\mathbb{E}_0(S_k)\|_p \le \sum_{\ell=1}^{k} \|\mathbb{E}_0(X_\ell)\|_p \text{ and } \|S_k - \mathbb{E}_k(S_k)\|_p \le \sum_{\ell=0}^{k-1} \|X_{-\ell} - \mathbb{E}_0(X_{-\ell})\|_p\,, \qquad (3.21)$$

we derive that for, any positive integers $N \geq n$,

$$\max_{1\leq \ell\leq n} \|R_\ell\|_p \ll \sum_{k=1}^{N} \|\mathbb{E}_0(X_k)\|_p + \sum_{k=0}^{N-1} \|X_{-k} - \mathbb{E}_0(X_{-k})\|_p + \\ + n^{1/2} \sum_{k\geq [N/2]} \frac{\|\mathbb{E}_0(X_k)\|_p}{k^{1/p}} + n^{1/2} \sum_{k\geq [N/2]} \frac{\|X_{-k} - \mathbb{E}_0(X_{-k})\|_p}{k^{1/p}} . \quad (3.22)$$

The lemma follows from (3.22) with $N = [n^{p/2}]$ by using Hölder's inequality (see the computations in the proof of Proposition 2.2 in [1]). $\square$

Proof of Lemma 3.2. Let m be a positive integer such that (3.3) is satisfied. We first write that

$$\|\mathbb{E}_{-nm}(M_n^2) - \mathbb{E}(M_n^2)\|_{p/2} \leq \|\mathbb{E}_{-nm}(S_n^2) - \mathbb{E}(S_n^2)\|_{p/2} \\ + 2\|\mathbb{E}_{-nm}(S_nR_n) - \mathbb{E}(S_nR_n)\|_{p/2} + 2\|R_n\|_p^2 .$$

By using Lemma 3.1, and since (3.3) holds, Lemma 3.2 will follow if we can prove that

$$\sum_{n\geq 1} \frac{1}{n^2(\log n)^{(t-1)p/2}} \|\mathbb{E}_{-nm}(S_nR_n)\|_{p/2}^{p/2} < \infty . \quad (3.23)$$

With this aim we shall prove the following inequality. For any non negative integer r and any positive integer u_n such that $u_n \leq n$, we have that

$$\|\mathbb{E}_{-r}(S_nR_n)\|_{p/2} \ll \sqrt{u_n}\big(\|\mathbb{E}_0(S_n)\|_2 + \|S_n - \mathbb{E}_n(S_n)\|_2\big) + \max_{k=\{n,n-u_n\}} \|R_k\|_p^2 + \\ + \sqrt{n}\big(\|\mathbb{E}_{-u_n}(S_n)\|_2 + \|S_n - \mathbb{E}_{n+u_n}(S_n)\|_2\big) + \max_{k=\{n,u_n\}} \|\mathbb{E}_{-r}(S_k^2) - \mathbb{E}(S_k^2)\|_{p/2} \\ + \sqrt{n}\Big(\sum_{k=1}^{n} \Big\| \sum_{|j|\geq k+n} P_0(X_j)\Big\|_2^2\Big)^{1/2} . \quad (3.24)$$

Let us show how, thanks to (3.24), the convergence (3.23) can be proven. Let us first consider the case where $2 < p < 4$. Notice that the following elementary claim is valid:

Claim 3.1. *If $\mathcal{F}$ and $\mathcal{G}$ are two σ-algebras such that $\mathcal{G} \subset \mathcal{F}$, then for any random variable X in $\mathbb{L}^q$ for $q \geq 1$, $\|X - \mathbb{E}(X|\mathcal{F})\|_q \leq 2\|X - \mathbb{E}(X|\mathcal{G})\|_q$.*

Starting from (3.24) with $r = nm$ and $u_n = n$, and using Claim 3.1, we derive that

$$\|\mathbb{E}_{-nm}(S_nR_n)\|_{p/2} \ll \|\mathbb{E}_{-nm}(S_n^2) - \mathbb{E}(S_n^2)\|_{p/2} + \sqrt{n}\big(\|\mathbb{E}_0(S_n)\|_2 + \|S_n - \mathbb{E}_n(S_n)\|_2\big) + \\ + \|R_n\|_p^2 + n \sum_{|j|\geq n} \|P_0(X_j)\|_2 .$$

This last inequality combined with condition (3.3) and Lemma 3.1 shows that (3.23) will be satisfied if we can prove that

$$\sum_{n\geq 1} \frac{n^{p/4}}{n^2(\log n)^{(t-1)p/2}} \big(\|\mathbb{E}_0(S_n)\|_2 + \|S_n - \mathbb{E}_n(S_n)\|_2\big)^{p/2} < \infty , \quad (3.25)$$

and

$$\sum_{n\geq 1}\frac{n^{p/2}}{n^2(\log n)^{(t-1)p/2}}\Big(\sum_{|j|\geq n}\|P_0(X_j)\|_2\Big)^{p/2}<\infty\,. \tag{3.26}$$

To prove (3.25), we use the inequalities (3.21) with $p=2$. Hence setting

$$a_\ell=\|\mathbb{E}_0(X_\ell)\|_2+\|X_{-\ell+1}-\mathbb{E}_0(X_{-\ell+1})\|_2\,, \tag{3.27}$$

and using Hölder's inequality, we derive that for any $\alpha<1$,

$$\sum_{n\geq 1}\frac{n^{p/4}}{n^2(\log n)^{(t-1)p/2}}\big(\|\mathbb{E}_0(S_n)\|_2+\|S_n-\mathbb{E}_n(S_n)\|_2\big)^{p/2}\ll\sum_{n\geq 1}\frac{n^{p/4}}{n^2(\log n)^{(t-1)p/2}}\Big(\sum_{\ell=1}^n a_\ell\Big)^{p/2}$$
$$\ll\sum_{n\geq 1}\frac{n^{p/4}n^{(1-\alpha)(p/2-1)}}{n^2(\log n)^{(t-1)p/2}}\sum_{\ell=1}^n \ell^{\alpha(p/2-1)}a_\ell^{p/2}\,.$$

Taking $\alpha\in](3p-8)/(2p-4),1[$ (this is possible since $p<4$) and changing the order of summation, we infer that (3.25) holds provided that (3.2) does. It remains to show that (3.26) is satisfied. Using Lemma 5.1 and the notation (3.27), we first observe that

$$\sum_{n\geq 1}\frac{n^{p/2}}{n^2(\log n)^{(t-1)p/2}}\Big(\sum_{|j|\geq n}\|P_0(X_j)\|_2\Big)^{p/2}\ll\sum_{n\geq 1}\frac{n^{p/2}}{n^2(\log n)^{(t-1)p/2}}\Big(\sum_{\ell\geq[n/2]}\ell^{-1/2}a_\ell\Big)^{p/2}\,.$$

Therefore by Hölder's inequality, it follows that for any $\alpha>1$,

$$\sum_{n\geq 1}\frac{n^{p/2}}{n^2(\log n)^{(t-1)p/2}}\Big(\sum_{|j|\geq n}\|P_0(X_j)\|_2\Big)^{p/2}\ll\sum_{n\geq 1}\frac{n^{p/2}n^{(1-\alpha)(p/2-1)}}{n^2(\log n)^{(t-1)p/2}}\sum_{\ell\geq[n/2]}\ell^{\alpha(p/2-1)}\ell^{-p/4}a_\ell^{p/2}\,.$$

Therefore taking $\alpha\in]1,2[$ and changing the order of summation, we infer that (3.25) holds provided that (3.2) does. This ends the proof of (3.23) when $p\in]2,4[$.

Now, we prove (3.23) when $p=4$. With this aim we start from (3.24) with $r=nm$ and $u_n=[\sqrt{n}]$. This inequality combined with condition (3.3), Lemma 3.1 and the arguments developed to prove (3.25) and (3.26) shows that (3.23) will be satisfied for $p=4$ if we can prove that

$$\sum_{n\geq 1}\frac{1}{n(\log n)^{2(t-1)}}\big(\|\mathbb{E}_{-[\sqrt{n}]}(S_n)\|_2+\|S_n-\mathbb{E}_{n+[\sqrt{n}]}(S_n)\|_2\big)^2<\infty\,, \tag{3.28}$$

and

$$\sum_{n\geq 2}\frac{1}{n^2(\log n)^{2(t-1)}}\big\|\mathbb{E}_{-nm}(S^2_{[\sqrt{n}]})-\mathbb{E}(S^2_{[\sqrt{n}]})\big\|_2^2<\infty\,. \tag{3.29}$$

We start by proving (3.28). With this aim, using the notation (3.27), we first write that

$$\|\mathbb{E}_{-[\sqrt{n}]}(S_n)\|_2+\|S_n-\mathbb{E}_{n+[\sqrt{n}]}(S_n)\|_2\leq\sum_{k=[\sqrt{n}]+1}^{n+[\sqrt{n}]}a_k\,.$$

Therefore by Cauchy-Schwarz's inequality

$$\sum_{n\geq 1}\frac{1}{n(\log n)^{2(t-1)}}\big(\|\mathbb{E}_{-[\sqrt{n}]}(S_n)\|_2+\|S_n-\mathbb{E}_{n+[\sqrt{n}]}(S_n)\|_2\big)^2 \ll \sum_{n\geq 1}\frac{\log n}{n(\log n)^{2(t-1)}}\sum_{k=[\sqrt{n}]+1}^{n+[\sqrt{n}]} k a_k^2$$
$$\ll \sum_{n\geq 1}\frac{1}{n}\sum_{k=[\sqrt{n}]+1}^{n+[\sqrt{n}]}\frac{k\log k}{(\log k)^{2(t-1)}}a_k^2 .$$

Changing the order of summation, this proves that (3.28) holds provided that (3.5) does. It remains to prove (3.29). With this aim, we set for any positive real x,

$$h([x]) = \big\|\mathbb{E}_{-m[x]}(S^2_{[x]}) - \mathbb{E}(S^2_{[x]})\big\|_2^2 ,$$

and we notice that, for any integer $n\geq 0$, $\big\|\mathbb{E}_{-nm}(S^2_{[\sqrt{n}]})-\mathbb{E}(S^2_{[\sqrt{n}]})\big\|_2^2 \leq h([\sqrt{n}])$. In addition, if $x\in[n,n+1[$, then $[\sqrt{n}]=[\sqrt{x}]$ or $[\sqrt{n}]=[\sqrt{x}]-1$. Therefore

$$\sum_{n\geq 3}\frac{1}{n^2(\log n)^{(t-1)p/2}}h([\sqrt{n}]) \ll \sum_{n\geq 3}h([\sqrt{n}])\int_{[n,n+1[}\frac{1}{x^2(\log x)^{(t-1)p/2}}dx$$
$$\ll \int_3^\infty \frac{1}{x^2(\log x)^{(t-1)p/2}}h([\sqrt{x}])dx + \int_3^\infty \frac{1}{x^2(\log x)^{(t-1)p/2}}h([\sqrt{x}]-1)dx$$
$$\ll \int_2^\infty \frac{1}{y^3(\log y)^{(t-1)p/2}}h([y])dy \ll \sum_{n\geq 2}\frac{1}{n^3(\log n)^{(t-1)p/2}}h(n)dy .$$

For the last inequality, we have used that if $y\in[n,n+1[$, then $[y]=n$. Therefore condition (3.3) implies (3.29). This ends the proof of (3.23) when $p=4$.

It remains to prove (3.24). With this aim, we start with the decomposition of R_n given in Proposition 5.1 of the appendix with $N=n$. Therefore setting

$$A_n := \sum_{k=1}^n\sum_{j\geq 2n+1}P_k(X_j) + \sum_{k=1}^n\sum_{j\geq n}P_k(X_{-j}) ,$$

we write that

$$R_n = \mathbb{E}_0(S_n)-\mathbb{E}_0(S_n)\circ T^n+\mathbb{E}_{-n}(S_n)\circ T^n+S_n-\mathbb{E}_n(S_n)-(\mathbb{E}_{2n}(S_n-\mathbb{E}_n(S_n))\circ T^{-n}-A_n . \quad (3.30)$$

Starting from (3.30) and noticing that

$$\|\mathbb{E}_{-r}(S_n(\mathbb{E}_{-n}(S_n)\circ T^n)\|_{p/2} \leq \|\mathbb{E}_0(S_n(\mathbb{E}_{-n}(S_n)\circ T^n)\|_{p/2} \leq \|\mathbb{E}_0(S_n)\|_p\|\mathbb{E}_0(S_{2n}-S_n)\|_p ,$$

and that $\mathbb{E}_{-r}(S_n(S_n-\mathbb{E}_n(S_n))=\mathbb{E}_{-r}((S_n-\mathbb{E}_n(S_n))^2)$, we first get

$$\|\mathbb{E}_{-r}(S_nR_n)\|_{p/2} \leq 2\|\mathbb{E}_0(S_n)\|_p^2 + \|S_n-\mathbb{E}_n(S_n)\|_p^2 + \|\mathbb{E}_{-r}(S_n\mathbb{E}_n(S_{2n}-S_n))\|_{p/2}$$
$$+ \|\mathbb{E}_{-r}(S_n\mathbb{E}_n(S_n\circ T^{-n}-\mathbb{E}_0(S_n\circ T^{-n})))\|_{p/2} + \big\|\mathbb{E}_{-r}(S_nA_n)\big\|_{p/2} . \quad (3.31)$$

Next, we use the following fact: if X and Y are two variables in $\mathbb{L}^p$ with $p\in[2,4]$, then for any integer u,

$$\|\mathbb{E}_u(XY)\|_{p/2} \leq \|\mathbb{E}_u(X^2)-\mathbb{E}(X^2)\|_{p/2} + \|Y\|_p^2 + \sqrt{\mathbb{E}(X^2)}\|Y\|_2 . \quad (3.32)$$

Indeed, it suffices to write that

$$\begin{aligned}
\|\mathbb{E}_u(XY)\|_{p/2} &\leq \|\mathbb{E}_u^{1/2}(X^2)\mathbb{E}_u^{1/2}(Y^2)\|_{p/2} \\
&\leq \||\mathbb{E}_u(X^2)-\mathbb{E}(X^2)|^{1/2}\mathbb{E}_u^{1/2}(Y^2)\|_{p/2} + (\mathbb{E}(X^2))^{1/2}\|\mathbb{E}_u^{1/2}(Y^2)\|_{p/2} \\
&\leq \|\mathbb{E}_u(X^2)-\mathbb{E}(X^2)\|_{p/2} + \|Y\|_p^2 + (\mathbb{E}(X^2))^{1/2}\|\mathbb{E}_u^{1/2}(Y^2)\|_{p/2}\,,
\end{aligned}$$

and to notice that, since $p \in [2,4]$, $\|\mathbb{E}_u^{1/2}(Y^2)\|_{p/2} \leq \|\mathbb{E}_u^{1/2}(Y^2)\|_2 = \|Y\|_2$. Therefore, starting from (3.31) and using (3.32) together with $\mathbb{E}(S_n^2) \ll n$, we infer that

$$\begin{aligned}
\|\mathbb{E}_{-r}(S_nR_n)\|_{p/2} \ll \|\mathbb{E}_0(S_n)\|_p^2 + \|S_n - \mathbb{E}_n(S_n)\|_p^2 + \|\mathbb{E}_{-r}(S_n\mathbb{E}_n(S_{2n}-S_n))\|_{p/2} \\
+ \|\mathbb{E}_{-r}(S_n\mathbb{E}_n(S_n\circ T^{-n} - \mathbb{E}_0(S_n\circ T^{-n})))\|_{p/2} \\
+ \big\|\mathbb{E}_{-r}(S_n^2) - \mathbb{E}(S_n^2)\big\|_{p/2} + \|A_n\|_p^2 + n^{1/2}\|A_n\|_2\,,
\end{aligned}$$

and since $\|\mathbb{E}_0(S_n)\|_p \leq \|R_n\|_p$, $\|S_n - \mathbb{E}_n(S_n)\|_p \leq 2\|R_n\|_p$ and $\|A_n\|_p \leq 8\|R_n\|_p$, we have overall that

$$\begin{aligned}
\|\mathbb{E}_{-r}(S_nR_n)\|_{p/2} \ll \|R_n\|_p^2 + \|\mathbb{E}_{-r}(S_n\mathbb{E}_n(S_{2n}-S_n))\|_{p/2} \\
+ \|\mathbb{E}_{-r}(S_n\mathbb{E}_n(S_n\circ T^{-n} - \mathbb{E}_0(S_n\circ T^{-n})))\|_{p/2} + \big\|\mathbb{E}_{-r}(S_n^2) - \mathbb{E}(S_n^2)\big\|_{p/2} + n^{1/2}\|A_n\|_2\,.
\end{aligned} \tag{3.33}$$

By orthogonality and by stationarity,

$$\begin{aligned}
\|A_n\|_2 &\leq \Big(\sum_{k=1}^n \Big\|\sum_{j\geq 2n+1} P_k(X_j)\Big\|_2^2\Big)^{1/2} + \Big(\sum_{k=1}^n \Big\|\sum_{j\geq n} P_k(X_{-j})\Big\|_2^2\Big)^{1/2} \\
&\leq \Big(\sum_{k=1}^n \Big\|\sum_{\ell\geq k+n} P_0(X_\ell)\Big\|_2^2\Big)^{1/2} + \Big(\sum_{k=1}^n \Big\|\sum_{\ell\geq k+n} P_0(X_{-\ell})\Big\|_2^2\Big)^{1/2}.
\end{aligned} \tag{3.34}$$

Now for any integer u_n such that $u_n \leq n$,

$$\begin{aligned}
\|\mathbb{E}_{-r}(S_n\mathbb{E}_n(S_{2n}-S_n))\|_{p/2} &\leq \|\mathbb{E}_{-r}((S_n - S_{n-u_n})\mathbb{E}_n(S_{2n}-S_n))\|_{p/2} \\
&\quad + \|\mathbb{E}_{-r}(S_{n-u_n}\mathbb{E}_n(S_{2n}-S_n))\|_{p/2} \\
&\ll \big\|\mathbb{E}_{-r}(S_{u_n}^2) - \mathbb{E}(S_{u_n}^2)\big\|_{p/2} + \|\mathbb{E}_0(S_n)\|_p^2 + \sqrt{u_n}\|\mathbb{E}_0(S_n)\|_2 \\
&\quad + \|\mathbb{E}_{-r}(S_{n-u_n}\mathbb{E}_n(S_{2n}-S_n))\|_{p/2}\,,
\end{aligned} \tag{3.35}$$

where for the last inequality we have used (3.32) together with $\mathbb{E}(S_{u_n}^2) \ll u_n$. Next, we write that

$$\begin{aligned}
\|\mathbb{E}_{-r}(S_{n-u_n}\mathbb{E}_n(S_{2n}-S_n))\|_{p/2} &\leq \|\mathbb{E}_{-r}((S_{n-u_n} - \mathbb{E}_{n-u_n}(S_{n-u_n}))\mathbb{E}_n(S_{2n}-S_n))\|_{p/2} \\
&\quad + \|\mathbb{E}_{-r}(\mathbb{E}_{n-u_n}(S_{n-u_n})\mathbb{E}_n(S_{2n}-S_n))\|_{p/2} \\
&\leq \|S_{n-u_n} - \mathbb{E}_{n-u_n}(S_{n-u_n})\|_p\|\mathbb{E}_0(S_n)\|_p + \|\mathbb{E}_{-r}(\mathbb{E}_{n-u_n}(S_{n-u_n})\mathbb{E}_{n-u_n}(S_{2n}-S_n))\|_{p/2} \\
&\leq \|S_{n-u_n} - \mathbb{E}_{n-u_n}(S_{n-u_n})\|_p^2 + \|\mathbb{E}_0(S_n)\|_p^2 + \|\mathbb{E}_{-r}(S_n\mathbb{E}_{n-u_n}(S_{2n}-S_n))\|_{p/2} \\
&\quad + \|\mathbb{E}_{-r}((S_n - S_{n-u_n})\mathbb{E}_{n-u_n}(S_{2n}-S_n))\|_{p/2}\,.
\end{aligned}$$

Therefore using (3.32), we infer that

$$\|\mathbb{E}_{-r}(S_{n-u_n}\mathbb{E}_n(S_{2n}-S_n))\|_{p/2} \ll \max_{k=\{n,n-u_n\}} \|R_k\|_p^2 + \sqrt{n}\|\mathbb{E}_{-u_n}(S_n)\|_2 + \max_{k=\{n,u_n\}} \|\mathbb{E}_{-r}(S_k^2) - \mathbb{E}(S_k^2)\|_{p/2}\,. \quad (3.36)$$

We deal now with the third term in the right-hand side of (3.33). With this aim, we first write that

$$\|\mathbb{E}_{-r}(S_n\mathbb{E}_n(S_n\circ T^{-n} - \mathbb{E}_0(S_n\circ T^{-n})))\|_{p/2} \le \|\mathbb{E}_{-r}(S_n\mathbb{E}_n(S_n\circ T^{-n} - \mathbb{E}_{u_n}(S_n\circ T^{-n})))\|_{p/2} + \|\mathbb{E}_{-r}(S_n\mathbb{E}_{u_n}(S_n\circ T^{-n} - \mathbb{E}_0(S_n\circ T^{-n})))\|_{p/2}. \quad (3.37)$$

By using (3.32) together with $\mathbb{E}(S_{u_n}^2) \ll n$, stationarity and the fact that $\|S_n - \mathbb{E}_{n+u_n}(S_n)\|_2 \le 2\|R_n\|_p$, we infer that

$$\|\mathbb{E}_{-r}(S_n\mathbb{E}_n(S_n\circ T^{-n} - \mathbb{E}_{u_n}(S_n\circ T^{-n})))\|_{p/2} \ll \|\mathbb{E}_{-r}(S_n^2) - \mathbb{E}(S_n^2)\|_{p/2} + \|R_n\|_p^2 + \sqrt{n}\|S_n - \mathbb{E}_{n+u_n}(S_n)\|_2\,. \quad (3.38)$$

On the other hand,

$$\|\mathbb{E}_{-r}(S_n\mathbb{E}_{u_n}(S_n\circ T^{-n} - \mathbb{E}_0(S_n\circ T^{-n})))\|_{p/2} \le \|\mathbb{E}_{-r}(S_{u_n}\mathbb{E}_{u_n}(S_n\circ T^{-n} - \mathbb{E}_0(S_n\circ T^{-n})))\|_{p/2} + \|\mathbb{E}_{-r}(\mathbb{E}_{u_n}(S_n - S_{u_n})\mathbb{E}_{u_n}(S_n\circ T^{-n} - \mathbb{E}_0(S_n\circ T^{-n}))\|_{p/2}\,.$$

We apply (3.32) to the first term of the right hand side together with $\mathbb{E}(S_{u_n}^2) \ll n$. Hence by stationarity and since $\|S_n - \mathbb{E}_n(S_n)\|_p \le 2\|R_n\|_p$, we derive that

$$\|\mathbb{E}_{-r}(S_{u_n}\mathbb{E}_{u_n}(S_n\circ T^{-n} - \mathbb{E}_0(S_n\circ T^{-n})))\|_{p/2} \ll \|\mathbb{E}_{-r}(S_{u_n}^2) - \mathbb{E}(S_{u_n}^2)\|_{p/2} + \\ + \|R_n\|_p^2 + \sqrt{u_n}\|S_n - \mathbb{E}_n(S_n)\|_2\,.$$

On the other hand, by stationarity,

$$\begin{aligned}\|\mathbb{E}_{-r}(\mathbb{E}_{u_n}(S_n - S_{u_n})\mathbb{E}_{u_n}(S_n\circ T^{-n} - \mathbb{E}_0(S_n\circ T^{-n})))\|_{p/2} &\\ \le \|\mathbb{E}_{u_n}(S_n - S_{u_n})\|_p\|\mathbb{E}_{u_n}(S_n\circ T^{-n} - \mathbb{E}_0(S_n\circ T^{-n}))\|_p &\\ \le \|\mathbb{E}_0(S_{n-u_n})\|_p\|S_n - \mathbb{E}_n(S_n))\|_p\,. &\\ \le \|\mathbb{E}_0(S_{n-u_n})\|_p^2 + \|R_n\|_p^2\,. &\end{aligned}$$

Therefore we get overall that

$$\|\mathbb{E}_{-r}(S_n\mathbb{E}_{u_n}(S_n\circ T^{-n} - \mathbb{E}_0(S_n\circ T^{-n})))\|_{p/2} \ll \|R_n\|_p^2 + \|\mathbb{E}_0(S_{n-u_n})\|_p^2 + \|\mathbb{E}_{-r}(S_{u_n}^2) - \mathbb{E}(S_{u_n}^2)\|_{p/2} + \sqrt{u_n}\|S_n - \mathbb{E}_n(S_n)\|_2\,. \quad (3.39)$$

Starting from (3.37) and taking into account (3.38) and (3.39), we get that

$$\|\mathbb{E}_{-r}(S_n\mathbb{E}_n(S_n\circ T^{-n} - \mathbb{E}_0(S_n\circ T^{-n})))\|_{p/2} \ll \sqrt{u_n}\|S_n - \mathbb{E}_n(S_n)\|_2 + \sqrt{n}\|S_n - \mathbb{E}_{n+u_n}(S_n)\|_2 + \max_{k=\{n,u_n\}}\|\mathbb{E}_{-r}(S_k^2) - \mathbb{E}(S_k^2)\|_{p/2} + \max_{k=\{n,n-u_n\}}\|R_k\|_p^2\,. \quad (3.40)$$

Finally, starting from (3.33) and considering (3.34), (3.35), (3.36) and (3.40), we conclude that (3.24) holds. □

4. Proof of Theorem 2.1

4.1. **Preparatory material.** Let us denote by E_u, E_e and E_s the S-stable vector spaces associated to the eigenvalues of S of modulus respectively larger than one, equal to one and smaller than one. Let d_u, d_e and d_s be their respective dimensions. Let $v_1, ..., v_d$ be a basis of $\mathbb{R}^d$ in which S is represented by a real Jordan matrix. Suppose that $v_1, ..., v_{d_u}$ are in E_u, $v_{d_u+1}, ..., v_{d_u+d_e}$ are in E_e and $v_{d_u+d_e+1}, ..., v_d$ are in E_s. We suppose moreover that $\det(v_1|v_2|\cdots|v_d) = 1$. Let us write $||\cdot||$ the norm on $\mathbb{R}^d$ given by

$$\left\| \sum_{i=1}^d x_i v_i \right\| = \max_{i=1,\dots,d} |x_i|$$

and $d_0(\cdot,\cdot)$ the metric induced by $||\cdot||$ on $\mathbb{R}^d$. Let also d_1 be the metric induced by d_0 on $\mathbb{T}^d$. We define now $B_u(\delta) := \{y \in E_u \ : \ ||y|| \le \delta\}$, $B_e(\delta) := \{y \in E_e \ : \ ||y|| \le \delta\}$ and $B_s(\delta) = \{y \in E_s \ : \ ||y|| \le \delta\}$. Let $|\cdot|$ be the usual euclidean norm on $\mathbb{R}^d$.

Let r_u be the spectral radius of $S^{-1}_{|E_u}$. For every $\rho_u \in (r_u, 1)$, there exists $K > 0$ such that, for every integer $n \ge 0$, we have

$$\forall h_u \in E_u, \quad ||S^n h_u|| \ge K \rho_u^{-n} ||h_u|| \tag{4.41}$$

and

$$\forall (h_e, h_s) \in E_e \times E_s, \quad ||S^n(h_e + h_s)|| \le K(1+n)^{d_e} ||h_e + h_s||. \tag{4.42}$$

Let $\rho_u \in (r_u, 1)$ and K satisfying (4.41) and (4.42). Let m_u, m_e, m_s be the Lebesgue measure on E_u (in the basis $v_1, ..., v_{d_u}$), E_e (in the basis $v_{d_u+1}, ..., v_{d_u+d_e}$) and E_s (in the basis $v_{d_u+d_e+1}, ..., v_d$) respectively. Observe that $d\lambda(h_u + h_e + h_s) = dm_u(h_u) dm_e(h_e) dm_s(h_s)$.

The properties satisfied by the filtration considered in [15, 12] and enabling the use of a martingale approximation method à la Gordin will be crucial here. Given a finite partition $\mathcal{P}$ of $\mathbb{T}^d$, we define the measurable partition $\mathcal{P}_0^\infty$ by :

$$\forall \bar{x} \in \mathbb{T}^d, \quad \mathcal{P}_0^\infty(\bar{x}) := \bigcap_{k \ge 0} T^k \mathcal{P}(T^{-k}(\bar{x}))$$

and, for every integer n, the σ-algebra $\mathcal{F}_n$ generated by

$$\forall \bar{x} \in \mathbb{T}^d, \quad \mathcal{P}_{-n}^\infty(\bar{x}) := \bigcap_{k \ge -n} T^k \mathcal{P}(T^{-k}(\bar{x})) = T^{-n}(\mathcal{P}_0^\infty(T^n(\bar{x})).$$

These definitions coincide with the ones of [12] applied to the ergodic toral automorphism T^{-1}. We obviously have $\mathcal{F}_n \subseteq \mathcal{F}_{n+1} = T^{-1}\mathcal{F}_n$. Let $r_0 > 0$ be such that $(h_u, h_e, h_s) \mapsto \overline{h_u + h_e + h_s}$ defines a diffeomorphism from $B_u(r_0) \times B_e(r_0) \times B_s(r_0)$ on its image in $\mathbb{T}^d$. Observe that, for every $\bar{x} \in \mathbb{T}^d$, on the set $\bar{x} + B_u(r_0) + B_e(r_0) + B_s(r_0)$, we have $d\bar{\lambda}(\bar{x} + \overline{h_u} + \overline{h_e} + \overline{h_s}) = dm_u(h_u) dm_e(h_e) dm_s(h_s)$.

Proposition 4.1 ([15, 12] applied to T^{-1}). *There exist some $Q > 0$, $K_0 > 0$, $\alpha \in (0,1)$ and some finite partition $\mathcal{P}$ of $\mathbb{T}^d$ whose elements are of the form $\sum_{i=1}^d I_i \overline{v_i}$ where the I_i are intervals with diameter smaller than* $\min(r_0, K)$ *such that, for almost every $\bar{x} \in \mathbb{T}^d$,*

1. *the local leaf $\mathcal{P}_0^\infty(\bar{x})$ of $\mathcal{P}_0^\infty$ containing $\bar{x}$ is a bounded convex set $\bar{x} + \overline{F(\bar{x})}$, with $0 \in F(\bar{x}) \subseteq E_u$, $F(\bar{x})$ having non-empty interior in E_u,*

2. *we have*

$$\mathbb{E}_n(f)(\bar{x}) = \frac{1}{m_u(S^{-n}F(T^n\bar{x}))}\int_{S^{-n}F(T^n\bar{x})} f(\bar{x}+\overline{h_u})\,dm_u(h_u), \tag{4.43}$$

3. *for every $\gamma > 0$, we have*

$$m_u(\partial(F(\bar{x}))(\gamma)) \le Q\gamma, \tag{4.44}$$

where

$$\partial F(\beta) := \{y \in F \ : \ d(y, \partial F) \le \beta\},$$

4. *for every $\mathbf{k} \in \mathbb{Z}^d \setminus \{0\}$, for every integer $n \ge 0$,*

$$\left|\mathbb{E}_{-n}(e^{2i\pi\langle \mathbf{k},\cdot\rangle})(\bar{x})\right| \le \frac{K_0}{m_u(F(T^{-n}(x)))}|\mathbf{k}|^{d_e+d_s}\alpha^n, \tag{4.45}$$

5. *for every $\beta \in (0,1)$,*

$$\exists L > 0,\ \forall n \ge 0,\ \bar{\lambda}(m_u(F(\cdot)) < \beta^n) \le L\beta^{n/d_u}. \tag{4.46}$$

Proof. The first item comes from Proposition II.1 of [12]. Item 2 comes from the formula given after Lemma II.2 of [12]. Item 3 follows from Lemma III.1 of [12] and from the fact that the numbers $a(\mathcal{P}_0^\infty(\cdot))$ considered in [12] are uniformly bounded. Item 4 comes from Proposition III.3 of [12] and from the uniform boundedness of $a(\mathcal{P}_0^\infty(\cdot))$. Item 5 comes from the proof of Proposition II.1 of [12]. □

According to the first item of Proposition 4.1 and to (4.41), there exists $c_u > 0$ such that, for almost every $\bar{x} \in \mathbb{T}^d$ and every $n \ge 1$, we have

$$\sup_{h_u \in S^{-n}F(T^n(\bar{x}))} |h_u| \le c_u \rho_u^n. \tag{4.47}$$

Proposition 4.2. *Let $p \ge 2$ and q be its conjugate exponent. Let $\theta > 0$ and $f : \mathbb{T}^d \to \mathbb{R}$ be a centered function with Fourier coefficients $(c_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ satisfying*

$$\sum_{|\mathbf{k}|\ge b} |c_{\mathbf{k}}|^q \le R\log^{-\theta}(b)\,. \tag{4.48}$$

Then

$$\|\mathbb{E}_0(f\circ T^n)\|_p = \|\mathbb{E}_{-n}(f)\|_p = O(n^{-\theta(p-1)/p})\,.$$

Proof. Recall first that $\mathbb{E}_0(f\circ T^n) = \mathbb{E}_{-n}(f)\circ T^n$. Let us consider α satisfying (4.45). Let $\beta := \alpha^{1/2}$, $\gamma := \max(\alpha^{p/2}, \beta^{1/d_u})$ and $\mathcal{V}_n := \left\{\bar{x} \in \mathbb{T}^d \ : \ m_u(F(T^{-n}(\bar{x})) \ge \beta^n\right\}$. Let $b(n) := \left\lceil \gamma^{-n/(2p(d+d_e+d_s))}\right\rceil$. Let us write

$$f = f_{1,n} + f_{2,n} \text{ where } f_{1,n} := \sum_{|\mathbf{k}|<b(n)} c_{\mathbf{k}} e^{2i\pi\langle\mathbf{k},\cdot\rangle} \text{ and } f_{2,n} := \sum_{|\mathbf{k}|\ge b(n)} c_{\mathbf{k}} e^{2i\pi\langle\mathbf{k},\cdot\rangle}. \tag{4.49}$$

We have

$$\begin{aligned}\int_{\mathcal{V}_n} |\mathbb{E}_{-n}(f_{1,n})|^p\,d\bar{\lambda} &\le \operatorname*{\mathbf{esssup}}_{\bar{x}\in\mathcal{V}_n}\Big(\sum_{|\mathbf{k}|\le b(n)} |c_{\mathbf{k}}|\left|\mathbb{E}_{-n}(e^{2i\pi\langle\mathbf{k},\cdot\rangle})(\bar{x})\right|\Big)^p \\ &\le \Big(\sum_{|\mathbf{k}|\le b(n)} |c_{\mathbf{k}}| K_0 \beta^{-n}|\mathbf{k}|^{d_e+d_s}\alpha^n\Big)^p,\end{aligned}$$

according to (4.45) and thanks to the definition of $\mathcal{V}_n$. Now, since $\beta = \alpha^{1/2}$, we get

$$\int_{\mathcal{V}_n} |\mathbb{E}_{-n}(f_{1,n})|^p \, d\bar{\lambda} \le 3^{dp} ||f||_1^p K_0^p \alpha^{\frac{np}{2}} (b(n))^{p(d+d_e+d_s)}.$$

Hence

$$\int_{\mathcal{V}_n} |\mathbb{E}_{-n}(f_{1,n})|^p \, d\bar{\lambda} = O(\gamma^n (b(n))^{p(d+d_e+d_s)}) = O(\gamma^{n/2}). \tag{4.50}$$

Moreover, thanks to (4.46), we have

$$\begin{aligned}\int_{\mathcal{V}_n^c} |\mathbb{E}_{-n}(f_{1,n})|^p \, d\bar{\lambda} &\le \bar{\lambda}(\mathcal{V}_n^c)\Big(\sum_{|\mathbf{k}|\le b(n)} |c_{\mathbf{k}}|\Big)^p \\ &= O((b(n))^{dp}\beta^{n/d_u}) = O((b(n))^{dp}\gamma^n) = O(\gamma^{n/2}). \end{aligned}\tag{4.51}$$

Since $p \ge 2$ and since $p/q = p-1$, thanks to (4.48), we have

$$\|\mathbb{E}_{-n}(f_{2,n})\|_p^p \le \|f_{2,n}\|_p^p \le \Big(\sum_{|\mathbf{k}|\ge b(n)} |c_{\mathbf{k}}|^q\Big)^{p/q} \le R^{p-1}(\log(b(n)))^{-\theta(p-1)} \ll n^{-\theta(p-1)}. \tag{4.52}$$

Combining (4.50), (4.51) and (4.52), the proposition follows. □

Proposition 4.3. *Under the assumptions of Proposition 4.2,*

$$\big\|\mathbb{E}_0(f\circ T^{-n}) - f\big\|_p = \|\mathbb{E}_n(f) - f\|_p = O(n^{-\theta(p-1)/p}).$$

Proof. We consider the decomposition (4.49) with $b(n)$ defined by $b(n) = \big[\rho_u^{-n/(2(d+1))}\big]$. We have

$$\begin{aligned}\|\mathbb{E}_n(f_{1,n}) - f_{1,n}\|_p &\le \|\mathbb{E}_n(f_{1,n}) - f_{1,n}\|_\infty \\ &\le \sum_{|\mathbf{k}|\le b(n)} |c_{\mathbf{k}}| \|\mathbb{E}_n(e^{2i\pi\langle\mathbf{k},\cdot\rangle}) - e^{2i\pi\langle\mathbf{k},\cdot\rangle}\|_\infty \\ &\le \sum_{|\mathbf{k}|\le b(n)} |c_{\mathbf{k}}| 2\pi|\mathbf{k}| c_u \rho_u^n,\end{aligned}$$

according to (4.43) and to (4.47). Therefore

$$\|\mathbb{E}_n(f_{1,n}) - f_{1,n}\|_p \ll (b(n))^{d+1}\rho_u^n \ll \rho_u^{n/2}. \tag{4.53}$$

Moreover, thanks to (4.48), we have

$$\begin{aligned}\|\mathbb{E}_n(f_{2,n}) - f_{2,n}\|_p^p &\le 2^p \|f_{2,n}\|_p^p \le 2^p\Big(\sum_{|\mathbf{k}|\ge b(n)} |c_{\mathbf{k}}|^q\Big)^{p/q} \\ &\le 2^p R^{p-1}(\log(b(n)))^{-\theta(p-1)} \ll n^{-\theta(p-1)}.\end{aligned}\tag{4.54}$$

Considering (4.53) and (4.54), the proposition follows. □

Proposition 4.4. *Let $p \in [2,4]$ and set $S_n(f) := \sum_{k=1}^n f\circ T^k$ with $f : \mathbb{T}^d \to \mathbb{R}$ be a centered function with Fourier coefficients satisfying (4.48) with $\theta > 0$ and*

$$\sum_{|\mathbf{k}|\ge b} |c_{\mathbf{k}}|^2 \le R\log^{-\beta}(b) \quad \textit{for some } \beta > 1. \tag{4.55}$$

Set

$$m := \Big[-\frac{4(d_e+d_s)\log(r)}{\log(\alpha)}\Big] + 1. \tag{4.56}$$

where r is the spectral radius of S. Then

$$\|\mathbb{E}_{-nm}(S_n^2(f)) - \mathbb{E}(S_n^2(f))\|_{p/2} \ll n^{2-2\theta(p-1)/p} + n^{(3-\beta)/2}\,.$$

Proof. Let $\beta := \alpha^{1/2}$, $\mathcal{V}_{nm} := \{\bar{x} \in \mathbb{T}^d \;:\; m_u(F(T^{-nm}(\bar{x})) \geq \beta^{nm}\}$, $\gamma := \max(\alpha^{p/8}, \beta^{1/d_u})$ and

$$b(n) := \left[\gamma^{n\,m/(p(2d+d_e+d_s))}\right]. \tag{4.57}$$

We consider the decomposition (4.49) with $b(n)$ defined by (4.57) and we set

$$S_{1,n}(f) := \sum_{k=1}^{n} f_{1,n} \circ T^k \;\text{ and } S_{2,n}(f) := \sum_{k=1}^{n} f_{2,n} \circ T^k\,.$$

First, we note that

$$\begin{aligned}
&\|\mathbb{E}_{-nm}(S_n^2(f)) - \mathbb{E}(S_n^2(f))\|_{p/2} \leq \|\mathbb{E}_{-nm}(S_{1,n}^2(f)) - \mathbb{E}(S_{1,n}^2(f))\|_{p/2} \\
&\quad + \|\mathbb{E}_{-nm}(S_{2,n}^2(f)) - \mathbb{E}(S_{2,n}^2(f))\|_{p/2} + 2\|\mathbb{E}_{-nm}(S_{1,n}(f)S_{2,n}(f)) - \mathbb{E}(S_{1,n}(f)S_{2,n}(f))\|_{p/2} \\
&\qquad \leq \|\mathbb{E}_{-nm}(S_{1,n}^2(f)) - \mathbb{E}(S_{1,n}^2(f))\|_{p/2} + 2\|S_{2,n}(f)\|_p^2 + 4\|\mathbb{E}_{-nm}(S_{1,n}(f)S_{2,n}(f))\|_{p/2}\,.
\end{aligned}$$

Next using (3.32), we get that

$$\begin{aligned}
\|\mathbb{E}_{-nm}(S_{1,n}(f)S_{2,n}(f))\|_{p/2} &\leq \|\mathbb{E}_{-nm}(S_{1,n}^2(f)) - \mathbb{E}(S_{1,n}^2(f))\|_{p/2} \\
&\quad + \|S_{2,n}(f)\|_p^2 + \|S_{1,n}(f)\|_2\|S_{2,n}(f)\|_2 \\
&\leq \|\mathbb{E}_{-nm}(S_{1,n}^2(f)) - \mathbb{E}(S_{1,n}^2(f))\|_{p/2} + 2\|S_{2,n}(f)\|_p^2 + \|S_n(f)\|_2\|S_{2,n}(f)\|_2\,.
\end{aligned}$$

By Propositions 4.2 and 4.3, (4.55) implies that

$$\sum_{n>0} \frac{\|\mathbb{E}_{-n}(f)\|_2}{n^{1/2}} < \infty \;\text{ and }\; \sum_{n>0} \frac{\|f - \mathbb{E}_n(f)\|_2}{n^{1/2}} < \infty\,,$$

which yields (3.6) with $p = 2$, and then $\|S_n(f)\|_2 \ll \sqrt{n}$. Therefore, we get overall that

$$\begin{aligned}
\|\mathbb{E}_{-nm}(S_n^2(f)) - \mathbb{E}(S_n^2(f))\|_{p/2} &\ll \|\mathbb{E}_{-nm}(S_{1,n}^2(f)) - \mathbb{E}(S_{1,n}^2(f))\|_{p/2} \\
&\quad + \|S_{2,n}(f)\|_p^2 + \sqrt{n}\|S_{2,n}(f)\|_2\,.
\end{aligned} \tag{4.58}$$

Since $p \geq 2$ and $p/q = p-1$, (4.48) implies that

$$\begin{aligned}
\|S_{2,n}(f)\|_p \leq n\|f_{2,n}\|_p &\leq n\Big(\sum_{|\mathbf{k}| \geq b(n)} |c_{\mathbf{k}}|^q\Big)^{1/q} \\
&\leq nR^{(p-1)/p}(\log(b(n)))^{-\theta(p-1)/p} \ll n^{1-\theta(p-1)/p}\,.
\end{aligned} \tag{4.59}$$

Similarly using (4.55), we get that

$$\|S_{2,n}(f)\|_2 \leq n\|f_{2,n}\|_2 \ll n^{1-\beta/2}\,. \tag{4.60}$$

We deal now with the first term in the right hand side of (4.58). With this aim, we first observe that, for any non negative integer ℓ, $e^{2i\pi\langle \mathbf{k}, T^\ell(\cdot)\rangle} = e^{2i\pi\langle {}^tS^\ell\mathbf{k}, \cdot\rangle}$, where ${}^tS^\ell$ is the transposed

matrix of S^ℓ. Therefore,

$$\int_{\mathcal{V}_{nm}} \left|\mathbb{E}_{-nm}(f_{1,n}.f_{1,n}\circ T^\ell) - \mathbb{E}(f_{1,n}.f_{1,n}\circ T^\ell)\right|^{p/2} d\bar\lambda$$
$$\le \operatorname*{\mathbf{esssup}}_{\bar x\in\mathcal{V}_{nm}} \Big(\sum_{|\mathbf{k}|,|\mathbf{m}|\le b(n):\mathbf{k}+{}^tS^\ell\mathbf{m}\neq 0} |c_\mathbf{k}||c_\mathbf{m}|\left|\mathbb{E}_{-nm}(e^{2i\pi\langle \mathbf{k}+{}^tS^\ell\mathbf{m},\cdot\rangle})(\bar x)\right|\Big)^{p/2}$$
$$\le \Big(\sum_{|\mathbf{k}|,|\mathbf{m}|\le b(n)} |c_\mathbf{k}||c_\mathbf{m}| K_0\beta^{-nm}|\mathbf{k}+{}^tS^\ell\mathbf{m}|^{d_e+d_s}\alpha^{nm}\Big)^{p/2},$$

according to (4.45) and to the definition of $\mathcal{V}_{nm}$. It follows that

$$\int_{\mathcal{V}_{nm}} \left|\mathbb{E}_{-nm}(f_{1,n}.f_{1,n}\circ T^\ell) - \mathbb{E}(f_{1,n}.f_{1,n}\circ T^\ell)\right|^{p/2} d\bar\lambda$$
$$\le \Big(\sum_{|\mathbf{k}|,|\mathbf{m}|\le b(n)} \|f\|_1^2 K_0\beta^{-nm}(|\mathbf{k}|+r^\ell|\mathbf{m}|)^{d_e+d_s}\alpha^{nm}\Big)^{p/2}$$
$$\ll \alpha^{\frac{nmp}{4}} r^{p\ell(d_e+d_s)/2}(b(n))^{p(2d+d_e+d_s)/2}\,.$$

Hence, since $\gamma\ge\alpha^{p/8}$, $m\ge 4(d_e+d_s)\log(r)/\log(1/\alpha)$, and according to the definition of $b(n)$, we have

$$\sup_{\ell\in\{0,\dots,n\}} \int_{\mathcal{V}_{nm}} \left|\mathbb{E}_{-nm}(f_{1,n}.f_{1,n}\circ T^\ell) - \mathbb{E}(f_{1,n}.f_{1,n}\circ T^\ell)\right|^{p/2} d\bar\lambda \ll \alpha^{3nmp/16} r^{pn(d_e+d_s)/2} \ll \gamma^{nm/2}. \tag{4.61}$$

Moreover, for any non negative integer ℓ,

$$\int_{\mathcal{V}^c_{nm}} \left|\mathbb{E}_{-nm}(f_{1,n}.f_{1,n}\circ T^\ell)\right|^{p/2} d\bar\lambda \le \bar\lambda(\mathcal{V}^c_{nm})\Big(\sum_{|\mathbf{k}|,|\mathbf{m}|\le b(n)} |c_\mathbf{k}||c_\mathbf{m}|\Big)^{p/2}$$
$$\ll (b(n))^{dp}\beta^{nm/d_u} \ll (b(n))^{dp}\gamma^{nm} \ll \gamma^{nm/2}\,, \tag{4.62}$$

according to (4.46) and to the definition of $b(n)$ and of γ. Combining (4.61) and (4.62), we then derive that

$$\|\mathbb{E}_{-nm}(S^2_{1,n}(f)) - \mathbb{E}(S^2_{1,n}(f))\|_{p/2}$$
$$\le 2\sum_{i=1}^{n}\sum_{j=0}^{n-i} \|\mathbb{E}_{-nm}(f_{1,n}\circ T^i f_{1,n}\circ T^{i+j}) - \mathbb{E}(f_{1,n}\circ T^i f_{1,n}\circ T^{i+j})\|_{p/2}$$
$$\le n^2 \sup_{\ell\in\{0,\dots,n\}} \|\mathbb{E}_{-nm}(f_{1,n}f_{1,n}\circ T^\ell) - \mathbb{E}(f_{1,n}f_{1,n}\circ T^\ell)\|_{p/2} \ll n^2\gamma^{nm/p}\,. \tag{4.63}$$

Considering (4.59), (4.60) and (4.63) in (4.58), the proposition follows. □

4.2. **End of the proof of Theorem 2.1.** Propositions 4.2 and 4.3 give (3.1) provided (2.2) is satisfied. Propositions 4.2 and 4.3 give (3.2) (when $p\in]2,4]$) and (3.5) (when $p=4$), provided (2.3) is satisfied. Finally, Proposition 4.4 gives (3.3) provided (2.2) and (2.3) are satisfied. The proof follows now from Theorem 3.1 when $p\in]2,4[$ and from Theorem 3.2 when $p=4$. □

5. Appendix

As in Section 3, let $P_k(X) = \mathbb{E}_k(X) - \mathbb{E}_{k-1}(X)$.

Lemma 5.1. *Let $p \in [2, \infty[$. Then, for any real $1 \le q \le p$ and any positive integer n,*

$$\sum_{k \ge 2n} \|P_0(X_k)\|_p^q \ll \sum_{k \ge n} \frac{\|\mathbb{E}_0(X_k)\|_p^q}{k^{q/p}} \quad \text{and} \quad \sum_{k \ge 2n} \|P_0(X_{-k})\|_p^q \ll \sum_{k \ge n} \frac{\|X_{-k} - \mathbb{E}_0(X_{-k})\|_p^q}{k^{q/p}} .$$

Proof. The first inequality is Lemma 5.1 in [1]. To prove the second one, we first consider the case $p > q$ and we follow the lines of the proof Lemma 5.1 in [1] with $P_k(X_0)$ replacing $P_{-k}(X_0)$. We get that

$$\sum_{k \ge 2n} \|P_0(X_{-k})\|_p^q \ll \sum_{k \ge n+1} k^{-\frac{q}{p}} \Big(\sum_{\ell \ge k} \|P_0(X_{-\ell})\|_p^p \Big)^{q/p} .$$

Now, we notice that, by the Rosenthal's inequality given in Theorem 2.12 of [8], there exists a constant c_p depending only on p such that

$$\begin{aligned} \sum_{\ell \ge k} \|P_0(X_{-\ell})\|_p^p &= \sum_{\ell \ge k} \|P_\ell(X_0)\|_p^p \\ &\le c_p \Big\| \sum_{\ell \ge k} P_\ell(X_0) \Big\|_p^p = c_p \|X_0 - \mathbb{E}_k(X_0)\|_p^p = c_p \|X_{-k} - \mathbb{E}_0(X_{-k})\|_p^p . \end{aligned} \tag{5.1}$$

Now when $p = q$, inequality (5.1) together with the fact that by Claim 3.1, for any integer k in $[n+1, 2n]$, $\|X_0 - \mathbb{E}_{2n}(X_0)\|_p^p \le 2^p \|X_0 - \mathbb{E}_k(X_0)\|_p^p$ imply the result. Indeed we have

$$\sum_{k \ge 2n} \|P_0(X_{-\ell})\|_p^p \le c_p \|X_0 - \mathbb{E}_{2n}(X_0)\|_p^p \ll \sum_{k=n+1}^{2n} k^{-1} \|X_0 - \mathbb{E}_k(X_0)\|_p^p . \quad \square$$

Proposition 5.1. *Let $p \in [1, \infty[$ and assume that*

$$\text{the series} \quad d_0 = \sum_{i \in \mathbb{Z}} P_0(X_i) \quad \text{converges in } \mathbb{L}^p. \tag{5.2}$$

Let $M_n := \sum_{i=1}^n d_0 \circ T^i$ and $R_n := S_n - M_n$. Then, for any positive integers n and N,

$$\begin{aligned} R_n = \mathbb{E}_0(S_n) - \mathbb{E}_0(S_N) \circ T^n + \mathbb{E}_{-n}(S_N) \circ T^n - \sum_{k=1}^n \sum_{j \ge n+N+1} P_k(X_j) \\ + S_n - \mathbb{E}_n(S_n) - (\mathbb{E}_{n+N}(S_N - \mathbb{E}_N(S_N)) \circ T^{-N} - \sum_{k=1}^n \sum_{j \ge N} P_k(X_{-j}) , \end{aligned}$$

and

$$\begin{aligned} \|R_n\|_p^{p'} \ll \|\mathbb{E}_0(S_n)\|_p^{p'} + \|\mathbb{E}_0(S_N)\|_p^{p'} + \|S_n - \mathbb{E}_n(S_n)\|_p^{p'} + \|S_N - \mathbb{E}_N(S_N)\|_p^{p'} \\ + \sum_{k=1}^n \Big\| \sum_{j \ge k+N} P_0(X_j) \Big\|_p^{p'} + \sum_{k=1}^n \Big\| \sum_{j \ge k+N} P_0(X_{-j}) \Big\|_p^{p'} , \end{aligned}$$

where $p' = \min(2, p)$.

Proof of Proposition 5.1. Notice first that the following decomposition is valid: for any positive integer n,

$$R_n = \sum_{k=1}^n \Big(X_k - \sum_{j=1}^n P_j(X_k) \Big) - \sum_{k=1}^n \sum_{j\geq n+1} P_k(X_j) - \sum_{k=1}^n \sum_{j=0}^\infty P_k(X_{-j}) = R_{n,1} + R_{n,2}\,, \quad (5.3)$$

where

$$R_{n,1} := \mathbb{E}_0(S_n) - \sum_{k=1}^n \sum_{j\geq n+1} P_k(X_j) \text{ and } R_{n,2} := S_n - \mathbb{E}_n(S_n) - \sum_{k=1}^n \sum_{j=0}^\infty P_k(X_{-j})\,. \quad (5.4)$$

Let N be a positive integer. According to item 1 of Proposition 2.1 in [1],

$$R_{n,1} = \mathbb{E}_0(S_n) - \mathbb{E}_n(S_{n+N} - S_n) + \mathbb{E}_0(S_{n+N} - S_n) - \sum_{k=1}^n \sum_{j\geq n+N+1} P_k(X_j)\,. \quad (5.5)$$

On an other hand, we write that $\sum_{j=0}^\infty P_k(X_{-j}) = \sum_{j=0}^{N-1} P_k(X_{-j}) + \sum_{j\geq N} P_k(X_{-j})$. Therefore

$$R_{n,2} = S_n - \mathbb{E}_n(S_n) - (\mathbb{E}_{n+N}(S_N - \mathbb{E}_N(S_N)) \circ T^{-N} - \sum_{k=1}^n \sum_{j\geq N} P_k(X_{-j})\,. \quad (5.6)$$

Starting from (5.3) and considering (5.5) and (5.6), the first part follows. We turn now to the second part of the proposition. Applying Burkholder's inequality and using stationarity, we obtain that there exists a positive constant c_p such that, for any positive integer n,

$$\Big\| \sum_{k=1}^n \sum_{j\geq n+N+1} P_k(X_j) \Big\|_p^{p'} \leq c_p \sum_{k=1}^n \Big\| \sum_{j\geq n+N+1} P_k(X_j) \Big\|_p^{p'} = c_p \sum_{k=1}^n \Big\| \sum_{j\geq N+k} P_0(X_j) \Big\|_p^{p'}\,, \quad (5.7)$$

and

$$\Big\| \sum_{k=1}^n \sum_{j\geq N} P_k(X_{-j}) \Big\|_p^{p'} \leq c_p \sum_{k=1}^n \Big\| \sum_{j\geq N} P_k(X_{-j}) \Big\|_p^{p'} = c_p \sum_{k=1}^n \Big\| \sum_{j\geq N+k} P_0(X_{-j}) \Big\|_p^{p'}\,. \quad (5.8)$$

The second part of the proposition follows from item 1 by taking into account stationarity and by considering the bounds (5.7) and (5.8). □

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